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The Domination Property for Efficiency in Locally Convex Spaces

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1. INTRODUCTION

One of the most important problems in vector optimization theory is to study the existence of efficient points of a set. This problem has been studied by many authors [6, 8–11, 19, 23]. Most of the previous existence results were obtained in normed linear spaces. Recently, Isac [18] studied the existence of efficient points for a kind of set in a locally convex space ordered by a nuclear cone. It is a more remarkable problem in vector optimization to study the domination property for efficiency. The domination property has been studied also by several authors [3, 18, 24, 26, 27]. In this paper, we give two concise sufficient conditions to guarantee that a set has the domination property in a locally convex space, and establish an existence theorem for super efficiency in more general settings. As an application, we give a strong Ekeland variational principle in a Banach space.

As observed by Kuhn and Tucker, some efficient points exhibit certain abnormal properties. To eliminate such anomalous efficient points, various concepts of proper efficiency have been introduced. It was originally introduced by Kuhn and Tucker [22] and Geoffrion [14], and later modified and formulated in a more general framework by Borwein [4], Hartly [15], Benson [2], Henig [16], and Borwein and Zhuang [7] (also see the references therein). Another important problem in vector optimization involves the density of the set of proper efficient points in the set of efficient points. This problem has been studied by Arrow, Barankin, and Blackwell [1], Hartley [15], Borwein [5], Jahn [20], Dauer and Gallagher [12], Petschke

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[25], Gallagher and Saleh [14], Zhuang [29], and Borwein and Zhuang [7]. Many previous density results were established for convex sets. In Section 4, for nonconvex sets, we prove a density theorem for super efficiency, which extends many existing density results. Moreover, we provide stability results.

2. PRELIMINARIES

For the sake of convenience, we make the following assumptions (unless specifically stated otherwise). Throughout this paper, X will always be a locally convex topological vector space (in brief, LCS) and a subset A of X is always assumed to be nonempty. $S \subset X$ is always assumed to be a pointed ($S \cap -S = \{0\}$) and convex cone which specifies a partial order in X as follows: $x, y \in X$, $x \leq_S y$ if $y - x \in S$. Let X^* denote the topological dual space of X. We associate a dual cone with S, denoting by S^+, in X^*, that is,

$$S^+ = \{ f \in X^* \mid f(x) \geq 0,\ x \in S \}.$$

The set of strictly positive functionals in S^+ is denoted by S^{+i}, that is,

$$S^{+i} = \{ f \in X^* \mid f(x) > 0,\ x \in S \text{ and } x \neq 0 \}.$$

Recall that a base of a cone S is a convex subset Θ of S such that

$$S = \operatorname{cone}(\Theta) = \{ t\theta \mid t \geq 0,\ \theta \in \Theta \} \qquad \text{and} \qquad 0 \notin \operatorname{cl}(\Theta).$$

It is obvious that S is pointed if S has a base.

We denote the set of efficient points of A with respect to the ordering cone S by $E(A, S)$, that is, $x \in E(A, S)$ iff

$$(A - x) \cap -S = \{0\}.$$

We say A has the domination property with respect to S if for each $x \in A$, there is $\bar{x} \in E(A, S)$ such that $x \geq_S \bar{x}$.

We denote the set of positive proper efficient points of A with respect to the ordering cone S by $\operatorname{Pos}(A, S)$, that is, $\bar{x} \in \operatorname{Pos}(A, S)$ iff there is $f \in S^{+i}$ such that

$$f(\bar{x}) = \inf\{ f(x) \mid x \in A \}.$$

It is clear that $\operatorname{Pos}(A, S) \subset E(A, S)$.

Recently, Borwein and Zhuang [7] introduced and studied the concept of super efficiency in a normal linear space, which is a very strong kind of proper efficiency. Let X be a normed linear space; $x \in A$ is said to be a super efficient point of a subset A of X with respect to the ordering cone S, written as $x \in SE(A, S)$, if there is a real number $M > 0$ such that

$$\mathrm{cl}(\mathrm{cone}(A - x)) \cap (B - S) \subset MB,$$

where B is the unit ball of X. It is known that $x \in SE(A, S)$ iff there is $M > 0$ such that

$$\|a - x\| \le M\|y\| \qquad \textit{whenever } a \in A \textit{ and } y \in X \textit{ with } a - x \le_S y. \quad (1)$$

The author [28] gave a generalization of super efficiency in a locally convex space.

Let X be a LCS. We say that $x \in A$ is a super efficient point of a subset A of X with respect to the ordering cone S, written as $x \in SE(A, S)$, if for each neighborhood V of 0, there is a neighborhood U of 0 such that

$$\mathrm{cl}(\mathrm{cone}(A - x)) \cap (U - S) \subset V.$$

It is clear that for any neighborhood W of 0, $\mathrm{cl}(\mathrm{cone}(A - x)) \subset \mathrm{cone}(A - x) + W$ and $(\mathrm{cone}(A - x) + W) \cap (U - S) \subset (\mathrm{cone}(A - x) \cap (-W + U - S)) + W$. It follows that $x \in SE(A, S)$ iff for each neighborhood V of 0, there is a neighborhood U of 0 such that

$$\mathrm{cone}(A - x) \cap (U - S) \subset V.$$

It is obvious that $SE(A, S)$ is a subset of $E(A, S)$.

3. DOMINATION PROPERTY AND EXISTENCE

Let S be a convex cone with a base Θ. The following notation will be used in this section:

$$\mathrm{int}_{\Theta}(S^+) = \{ f \in S^+ | \inf\{ f(\theta) \mid \theta \in \Theta\} > 0\}.$$

By the separation theorem, $\mathrm{int}_{\Theta}(S^+) \neq \emptyset$. Clearly,

$$S^{+i} \supset \mathrm{int}_{\Theta}(S^+) \supset S^+ + \mathrm{int}_{\Theta}(S^+).$$

PROPOSITION 3.1. *Let X be a normed linear space and $S \subset X$ a convex cone with a bounded base Θ. Then* $\mathrm{int}_{\Theta}(S^+) = \mathrm{int}(S^+)$, *where* $\mathrm{int}(S^+)$ *is the interior of S^+ in X^*.*

Proof. Let $\delta = \inf\{\|\theta\| \mid \theta \in \Theta\}$. Then $\delta > 0$. For each $f \in \mathrm{int}(S^+)$, there is $\varepsilon > 0$ such that whenever $g \in X^*$ and $\|g\| \le \varepsilon, f + g \in S^+$. Take

$\theta_\varepsilon \in \Theta$ such that

$$f(\theta_\varepsilon) < \inf\{f(\theta) \mid \theta \in \Theta\} + \frac{\varepsilon\delta}{2}. \tag{2}$$

By the Hahn–Banach theorem, there is $g_\varepsilon \in X^*$ such that $\|g_\varepsilon\| = \varepsilon$ and $g_\varepsilon(\theta_\varepsilon) = -\varepsilon\|\theta_\varepsilon\|$. Hence $f + g_\varepsilon \in S^+$. So, $f(\theta_\varepsilon) - \varepsilon\|\theta_\varepsilon\| = (f + g_\varepsilon)(\theta_\varepsilon) \geq 0$, that is, $f(\theta_\varepsilon) \geq \varepsilon\|\theta_\varepsilon\| \geq \varepsilon\delta$. By (2), $\inf\{(f(\theta) \mid \theta \in \Theta\} \geq \varepsilon\delta/2 > 0$. This implies that $f \in \operatorname{int}_\Theta(S^+)$. Therefore, $\operatorname{int}(S^+) \subset \operatorname{int}_\Theta(S^+)$. By the boundedness of Θ, it is easy to verify that $\operatorname{int}(S^+) \supset \operatorname{int}_\Theta(S^+)$. Hence $\operatorname{int}(S^+) = \operatorname{int}_\Theta(S^+)$.

It is known that $\operatorname{int}(S^+) \neq \emptyset$ iff S has a bounded base (see [21]).

LEMMA 3.1. *Let X be a LCS and $S \subset X$ a convex cone with a bounded base Θ. The following assertions hold*:

(i) *if $x_1 \geq_S x_2 \geq_S \cdots \geq_S x_n \geq_S \cdots$, and there is $f \in \operatorname{int}_\Theta(S^+)$ such that the scalar sequence $\{f(x_n)\}$ is lower bounded, then $\{x_n\}$ is a Cauchy sequence*,

(ii) *if $\{x_n\} \subset S$ and there is $f \in \operatorname{int}_\Theta(S^+)$ such that $f(x_n) \to 0$, then $x_n \to 0$.*

Proof. (i) Let $\alpha = \inf\{f(\theta) \mid \theta \in \Theta\}$. Then $\alpha > 0$. For each neighborhood V of 0, there is $\delta > 0$ such that for each $0 \leq t \leq \delta$, $t\Theta \subset V$ (because Θ is bounded). By condition (i) the scalar sequence $\{f(x_n)\}$ is convergent. Hence there is an index n_0 such that whenever $n \geq m \geq n_0$, $f(x_m - x_n) < \alpha\delta$. Since whenever $n \geq m$, $x_m - x_n \geq_S 0$, there are $\lambda_{mn} \geq 0$ and $\theta_{mn} \in \Theta$ such that $x_m - x_n = \lambda_{mn}\theta_{mn}$. Therefore, whenever $n \geq m \geq n_0$,

$$\alpha\delta > f(x_m - x_n) = \lambda_{mn} f(\theta_{mn}) \geq \lambda_{mn}\alpha,$$

and so, $\lambda_{mn} < \delta$. Hence, whenever $n \geq m \geq n_0$, $x_m - x_n = \lambda_{mn}\theta_{mn} \in V$. It follows that $\{x_n\}$ is a Cauchy sequence.

(ii) By $x_n \in S$, there are $\lambda_n \geq 0$ and $\theta_n \in \Theta$ such that $x_n = \lambda_n\theta_n$. Hence,

$$0 \leq \lambda_n\alpha \leq \lambda_n f(\theta_n) = f(x_n).$$

From $f(x_n) \to 0$, one has $\lambda_n \to 0$. Therefore, $x_n = \lambda_n\theta_n \to 0$ (because Θ is bounded).

Now we are able to prove the main result of this section.

THEOREM 3.1. *Let X be a LCS, $S \subset X$ a closed convex cone, and $A \subset X$ a sequentially complete set. Suppose that S has a bounded base Θ and there is*

$f \in \operatorname{int}_{\Theta}(S^+)$ such that f is lower bounded on A. Then A has the domination property with respect to S. Hence, $E(A, S) \neq \varnothing$.

Proof. For each $x_0 \in A$,

$$\inf\{f(x) \mid x \in A \cap (x_0 - S)\} \geq \inf\{f(x) \mid x \in A\} > -\infty.$$

Hence, for each $\varepsilon > 0$, there is $x_\varepsilon \in A \cap (x_0 - S)$ such that

$$f(x_\varepsilon) < \inf\{f(x) \mid x \in A \cap (x_0 - S)\} + \varepsilon. \tag{3}$$

It is clear that

$$x_0 \geq_S x_\varepsilon. \tag{4}$$

Using (3) and (4), we can construct inductively a sequence $\{x_n\}$ in A such that

(1) $x_0 \geq_S x_1 \geq_S \cdots \geq_S x_n \geq_S \cdots$.

(2) $f(x_n) < \inf\{f(x) \mid x \in A \cap (x_{n-1} - S)\} + 1/n$, $n = 1, 2, \ldots$.

By (i) of Lemma 3.1, $\{x_n\}$ is a Cauchy sequence in A. Since A is sequentially complete, there is $\bar{x} \in A$ such that $x_n \to \bar{x}$. By (1), and since S is a closed cone, $x_n \geq_S \bar{x}$, for all n. We claim that $\bar{x} \in E(A, S)$. Indeed, if $\bar{x} \notin E(A, S)$, there would be $\bar{y} \in A$ such that $\bar{x} \geq_S \bar{y}$ and $\bar{x} \neq \bar{y}$. This implies that $f(\bar{x}) > f(\bar{y})$ and $x_n \geq_S \bar{y}$, for all n. Hence $\bar{y} \in A \cap (x_n - S)$. By (2),

$$f(\bar{y}) \geq \inf\{f(x) \mid x \in A \cap (x_n - S)\} > f(x_{n+1}) - \frac{1}{n}.$$

Let $n \to \infty$. One has $f(\bar{y}) \geq f(\bar{x})$, a contradiction.

Remark. In a normed linear space, the sequential completeness of a set A is equivalent to the completeness of A. However, in the setting of a locally convex space, the sequential completeness of a set A is weaker than the completeness of A. For example, let X be a normed linear space and w denote the weak topology of X. Then (X, w) is a locally convex space. It is known that a bounded subset A of X is complete with respect to the weak topology w iff A is compact with respect to w. Take $X = l^1$ and $A = \{x \in X \mid \|x\| \leq 1\}$. Then A is not compact with respect to w, and so, A is not complete with respect to w. But, A is sequentially complete with respect to w (because a sequence $\{x_n\}$ in l^1 is weakly convergent iff $\{x_n\}$ is convergent).

The following corollaries follow immediately from Theorem 3.1.

COROLLARY 3.1. *Let X be a LCS, $S \subset X$ a closed convex cone with a bounded base, and $A \subset X$ a complete and bounded set. Then A has the domination property with respect to S.*

COROLLARY 3.2. *Let X be a LCS, $S \subset X$ a closed convex cone with a bounded base Θ, and $A \subset X$ a set. Suppose that there are $x_0 \in A$ and $f \in \mathrm{int}_\Theta(S^+)$ such that $A \cap (x_0 - S)$ is complete and f is lower bounded on $A \cap (x_0 - S)$. Then $E(A, S) \neq \emptyset$.*

THEOREM 3.2. *Let X be a LCS, S be a closed convex cone in X with a bounded base Θ, and A be a closed set in X. Assume that Θ is sequentially complete and there is $f \in \mathrm{int}_\Theta(S^+)$ such that f is lower bounded on A. Then A has the domination property with respect to S.*

Proof. By Corollary 3.2, it suffices to show that for each $x \in A$, $A \cap (x - S)$ is sequentially complete. For any Cauchy sequence $\{a_n\}$ in $A \cap (x - S)$, there is a sequence $\{t_n\}$ in R and a sequence $\{\theta_n\}$ in Θ such that for each n, $t_n \geq 0$, and

$$a_n = x - t_n \theta_n, \tag{5}$$

(i) If $t_n \to 0$, by the boundedness of Θ and (5), $a_n \to x \in A \cap (x - S)$.

(ii) If $\{t_n\}$ does not converge to 0, extracting a subsequence if necessary, we can assume that $t_n \to t_0 > 0$. Hence $\{(1/t_n)(x - a_n)\}$ is a Cauchy sequence, that is, $\{\theta_n\}$ is a Cauchy sequence in Θ. By the sequential completeness of Θ, there is $\theta \in \Theta$ such that $\theta_n \to \theta$. We claim that $t_0 \neq +\infty$. Indeed, from $f(x) - t_n f(\theta_n) = f(a_n)$, we have

$$t_n = \frac{f(x) - f(a_n)}{f(\theta_n)} \leq \frac{f(x) - \inf\{f(a) \mid a \in A\}}{\inf\{f(\theta) \mid \theta \in \Theta\}}.$$

It follows that $\{t_n\}$ is a bounded scalar sequence. Therefore, $t_0 \neq +\infty$. By (5), and since both A and S are closed, $a_n \to x - t_0\theta \in A \cap (x - S)$.

Parts (i) and (ii) imply that $A \cap (x - S)$ is sequentially complete.

The following results involve the existence of super efficient points of a set.

PROPOSITION 3.2. *Let X be a LCS, $S \subset X$ be a convex cone with a bounded base Θ, and $A \subset X$ be a set. Assume that there are $f \in \mathrm{int}_\Theta(S^+)$ and $\bar{x} \in A$ such that $f(\bar{x}) = \inf\{f(x) \mid x \in A\}$. Then $\bar{x} \in SE(A, S)$.*

Proof. Let $\mathbf{N}(0)$ denote the family of all neighborhoods of 0 in X. Suppose that $\bar{x} \notin SE(A, S)$. Then there would be $V_0 \in \mathbf{N}(0)$ such that for each $U \in \mathbf{N}(0)$, $\mathrm{cone}(A - \bar{x}) \cap (U - S)$ is not a subset of V_0. Hence, for each $U \in \mathbf{N}(0)$, there are $x_U \in U$, $t_U \geq 0$, and $\theta_U \in \Theta$ such that

$$x_U - t_U\theta_U \in \operatorname{cone}(A - \bar{x}) \tag{6}$$

and

$$x_U - t_U\theta_U \notin V_0. \tag{7}$$

Pick a neighborhood V_1 of 0 such that $V_1 - V_1 \subset V_0$. It is clear that the net $\{x_U\}_{U \in \mathbf{N}(0)}$ converges to 0 in X. Without loss of generality, we can assume $\{x_U\}_{U \in \mathbf{N}(0)} \subset V_1$. By (7) and $V_1 - V_1 \subset V_0$, for each $U \in \mathbf{N}(0)$, $t_U\theta_U \notin V_1$. From the boundedness of Θ, we can find $t_0 > 0$ such that whenever $0 \leq t < t_0$, $t\Theta \subset V_1$. It follows that for each $U \in \mathbf{N}(0)$, $t_U \geq t_0$. By (6) and $f(\bar{x}) = \inf\{f(x) \,|\, x \in A\}$, $f(x_U) - t_U f(\theta_U) \geq 0$, and so,

$$f(x_U) \geq t_U f(\theta_U) \geq t_0 \inf\{f(\theta) \mid \theta \in \Theta\}.$$

Hence,

$$0 = \lim_U f(x_U) \geq t_o \inf\{f(\theta) \mid \theta \in \Theta\}.$$

This contradicts $f \in \operatorname{int}_\Theta(S^+)$.

PROPOSITION 3.3. *Let X be a LCS, S be a convex cone in X with a base Θ, and A be a convex set in X. Assume that $\bar{x} \in SE(A, S)$. Then there is $f \in \operatorname{int}_\Theta(S^+)$ such that $f(\bar{x}) = \inf\{f(x) \,|\, x \in A\}$.*

Proof. Since Θ is a base of S, there is a convex neighborhood V of 0 such that

$$(-\Theta) \cap V = \emptyset. \tag{8}$$

By $\bar{x} \in SE(A, S)$, there is an open convex neighborhood U of 0 such that $U \subset \frac{1}{2}V$ and $\operatorname{cone}(A - \bar{x}) \cap (U - S) \subset \frac{1}{2}V$. Hence

$$\operatorname{cone}(A - \bar{x}) \cap (U - \Theta) \subset \frac{1}{2}V \cap (U - \Theta). \tag{9}$$

By (8) and $U \subset \frac{1}{2}V$, $\frac{1}{2}V \cap (U - \Theta) = \emptyset$. This and (9) imply that

$$\operatorname{cone}(A - \bar{x}) \cap (U - \Theta) = \emptyset.$$

By the separation theorem, there is $f \in X^*$, $f \neq 0$ such that

$$\inf\{f(x) \,|\, x \in \operatorname{cone}(A - \bar{x})\} \geq \sup\{f(x) \,|\, x \in U - \Theta\}.$$

It follows that $\inf\{f(x) \,|\, x \in \operatorname{cone}(A - \bar{x})\} = 0$. Hence $f(\bar{x}) = \inf\{f(x) \,|\, x \in A\}$ and

$$0 \geq \sup\{f(x) \,|\, x \in U - \Theta\}. \tag{10}$$

It suffices to show that $f \in \operatorname{int}_{\Theta}(S^+)$. By $f \neq 0$, and since U is a neighborhood of 0, there is $u_0 \in U$ such that $f(u_0) > 0$. For each $\theta \in \Theta$, by (10), $f(u_0 - \theta) \leq 0$, that is, $f(u_0) \leq f(\theta)$. Hence $0 < f(u_0) \leq \inf\{f(\theta) \mid \theta \in \Theta\}$. This implies $f \in \operatorname{int}_{\Theta}(S^+)$.

THEOREM 3.3. *Let X be a normed linear space, $S \subset X$ a closed convex cone with a bounded base, and $A \subset X$ a complete set. Suppose that there is $f \in \operatorname{int}(S^+)$ such that f is lower bounded on A. Then $SE(A, S) \neq \emptyset$ (cf. Proposition* 2.4 *and Theorem* 2.7 *in* [7]).

Proof. Let Θ be a bounded base of S, and $\delta = \inf\{\|\theta\| \mid \theta \in \Theta\}$. Then $\delta > 0$. By Proposition 3.1, $f \in \operatorname{int}_{\Theta}(S^+)$, that is, $\alpha = \inf\{f(\theta) \mid \theta \in \Theta\} > 0$. Take $0 < \varepsilon < \delta$ such that $\sup\{|f(x)| \,|\, x \in \varepsilon B\} < \alpha/2$, where B is the unit ball of X. Then $0 \notin \operatorname{cl}(\Theta + \varepsilon B)$ and $\inf\{f(x) \,|\, x \in \Theta + \varepsilon B\} \geq a/2$. This implies that $\operatorname{cl}(\Theta + \varepsilon B)$ is a bounded base of $S_{\varepsilon}(\Theta) = \operatorname{cl}(\operatorname{cone}(\Theta + \varepsilon B))$ and $f \in \operatorname{int}_{\operatorname{cl}(\Theta + \varepsilon B)}((S_{\varepsilon}(\Theta))^+)$. By Theorem 3.1 and [7, Corollary 2.6], $\emptyset \neq E(A, S_{\varepsilon}(\Theta)) \subset SE(A, S)$.

The following corollary may be regarded as a strong Ekeland variational principle in a Banach space.

COROLLARY 3.3. *Let X be a Banach space and $f: X \to R \cup \{+\infty\}$ a proper lower semicontinuous and lower bounded function. Then for any $\varepsilon > 0$, there are $\bar{x} \in X$ and $M > 0$ such that*

(i) $f(x) + \varepsilon\|x - \bar{x}\| > f(\bar{x})$, *for each $x \in X$ and $x \neq \bar{x}$.*

(ii) $\|x - \bar{x}\| \leq M|f(x) + \varepsilon\|x - \bar{x}\| - f(\bar{x})|$ *for each $x \in X$.*

(iii) $|f(x) - f(\bar{x})| \leq M|f(x) + \varepsilon\|x - \bar{x}\| - f(\bar{x})|$, *for each $x \in X$.*

Proof. Let $E = X \times R$, $S = \{(x, t) \in E \mid \varepsilon\|x\| \leq t\}$, and $x^*(x, t) = t$, for all $(x, t) \in E$. Then S is a convex cone with a bounded base $\Theta = \{(x, 1) \,|\, x \in X,\ \varepsilon\|x\| \leq 1\}$ and $x^* \in \operatorname{int}_{\Theta}(S^+)$. Let $A = \{(x, t) \in E \,|\, f(x) \leq t\}$. Since f is lower semicontinuous and lower bounded, A is a closed set in E and x^* is lower bounded on A. By Theorem 3.3, there is $(\bar{x}, \bar{t}) \in A$ such that $(\bar{x}, \bar{t}) \in SE(A, S)$. It is not hard to verify that $\bar{t} = f(\bar{x})$. From $SE(A, S) \subset E(A, S)$, we have $(A - (\bar{x}, f(\bar{x}))) \cap S = \{0\}$. Hence, for each $x \in X$ with $x \neq \bar{x}$,

$$f(x) + \varepsilon\|x - \bar{x}\| > f(\bar{x}).$$

For $x \in X$ with $f(x) = +\infty$, it is clear that (ii) and (iii) hold. For $x \in X$ with $f(x) < +\infty$, let $r = f(x) + \varepsilon\|x - \bar{x}\| - f(\bar{x})$. It is easy to verify that

$$(x, f(x)) - (\bar{x}, f(\bar{x})) \leq_S (0, r).$$

By (1), there is $M > 0$ (does not depend upon x) such that

$$\|(x, f(x) - (\bar{x}, f(\bar{x}))\| \leq M\|(0, r)\| = M|r|.$$

This implies that

$$\|x - \bar{x}\| \leq M\big| f(x) + \varepsilon\|x - \bar{x}\| - f(\bar{x})\big|$$

and

$$\big| f(x) - f(\bar{x})\big| \leq M\big| f(x) + \varepsilon\|x - \bar{x}\| - f(\bar{x})\big|.$$

Remark. Corollary 3.3 explains that $\bar{x}$ is not only the unique minimizer of the perturbed function $g_\varepsilon : X \to R \cup \{+\infty\}$ defined by $g_\varepsilon(x) = f(x) + \varepsilon\|x - \bar{x}\|$, but also $x_n \to \bar{x}$ and $f(x_n) \to f(\bar{x})$ whenever $\{x_n\} \subset X$ and $g_\varepsilon(x_n) \to g_\varepsilon(\bar{x})$.

4. DENSITY RESULTS FOR SUPER EFFICIENCY

The following theorem is the main result of this section, which generalizes many previous density results.

THEOREM 4.1. *Let X be a LCS, $S \subset X$ a convex cone with a closed bounded base Θ, and $A \subset X$ a weakly compact set, then* $\operatorname{cl}(SE(A, S)) \supset E(A, S)$.

Proof. It is sufficient to show that for each $\bar{x} \in E(A, S)$ and each neighborhood V of 0,

$$(\bar{x} + V) \cap SE(A, S) \neq \varnothing.$$

Since Θ is a base of S, $0 \notin \operatorname{cl}(\Theta)$. By the separation theorem, there is $f \in X^*$ such that $\alpha = \inf\{f(\theta) \mid \theta \in \Theta\} > 0$. Pick a convex neighborhood U_0 of 0 such that $U_0 \subset \{x \in X \mid |f(x)| < \alpha/2\}$. Hence

$$\inf\{f(x) \mid x \in U_0 + \Theta\} > 0. \tag{11}$$

Let $\mathbf{N}(0)$ be the family of all neighborhoods of 0 in X and $\mathbf{N}_0 = \{U \in \mathbf{N}(0) \mid U$ is a convex subset of $U_0\}$. We claim that there is $U_1 \in \mathbf{N}_0$ such that

$$A_\Theta = (A - \bar{x}) \cap -\operatorname{cl}\big(S_{U_1}(\Theta)\big) \subset V,$$

where $S_U = \operatorname{cone}(U + \Theta)$. Indeed, suppose that it were false. Then for each $U \in \mathbf{N}_0$, $(A - \bar{x}) \cap -\operatorname{cl}(S_U(\Theta))$ would be not a subset of V. Hence, for each $U \in \mathbf{N}_0$ there is $a_U \in A$ such that $a_U - \bar{x} \in -\mathrm{cl}(S_U(\Theta))$ and

$$a_U - \bar{x} \notin V. \tag{12}$$

Hence $(a_U - \bar{x} + U) \cap - S_U(\Theta) \neq \emptyset$, and so, there are $x_U \in U$, $y_U \in U$, $\theta_U \in \Theta$, and $t_U \geq 0$ such that

$$a_U - \bar{x} + x_U = -t_U(y_U + \theta_U). \tag{13}$$

It is clear that both the net $\{x_U\}_{U \in \mathbf{N}_0}$ and the net $\{y_U\}_{U \in \mathbf{N}_0}$ converge to 0 in X. It follows from (12), (13), and the boundedness of Θ that each subnet of the net $\{t_U\}_{U \in \mathbf{N}_0}$ does not converge to 0 (i.e., $\liminf_U t_U > 0$). Extracting a subnet if necessary we can assume that $t_U \to t > 0$. Since A is weakly compact, without loss of generality we can assume that $\{a_U\}_{U \in \mathbf{N}_0}$ weakly converges to some a in A. By (13),

$$-\theta_U = \frac{1}{t_U}(a_U - \bar{x} + x_U) + y_U \xrightarrow{w} \frac{1}{t}(a - \bar{x}). \tag{14}$$

Since each closed convex set in a LCS is weakly closed, Θ is a weakly close set. By (14), $(1/t)(a - \bar{x}) \in -\Theta$. This contradicts $\bar{x} \in E(A, S)$. By (11) and $U_1 \subset U_0$, $f \in (S_{U_1}(\Theta))^{+i}$. From the convexity of U_1 and Θ, we have that $-\mathrm{cl}(S_{U_1}(\Theta))$ is a weakly close set in X. This and the weak compactness of A imply that A_Θ is a weakly compact subset of A. Hence there is $x_0 \in A$ such that $x_0 - \bar{x} \in A_\Theta$ and

$$f(x_0 - \bar{x}) = \inf\{f(x) \mid x \in A_\Theta\}.$$

By $f \in (S_{U_1}(\Theta))^{+i}$, $x_0 - \bar{x} \in \mathrm{Pos}(A_\Theta, S_{U_1}(\Theta)) \subset E(A_\Theta, S_{U_1}(\Theta))$. It is easy to verify that $x_0 - \bar{x} \in E(A - \bar{x}, S_{U_1}(\Theta))$, and so, $x_0 \in E(A, S_{U_1}(\Theta))$. By $x_0 - \bar{x} \in A_\Theta \subset V$, it suffices to show that $x_0 \in SE(A, S)$.

Suppose that $x_0 \notin SE(A, S)$. Then there would be $V_0 \in \mathbf{N}(0)$ such that for each $U \in \mathbf{N}(0)$, $\mathrm{cone}(A - x_0) \cap (U - S)$ is not a subset of V_0. Hence, for each $U \in \mathbf{N}(0)$, there are $b_U \in A$, $z_U \in U$, $\theta_U \in \Theta$, $r_U \geq 0$, and $s_U \geq 0$ such that

$$r_U(b_U - x_0) = z_U - s_U\theta_U \notin V_0. \tag{15}$$

Clearly, $r_U > 0$ for all $U \in \mathbf{N}(0)$. Pick a neighborhood V_1 of 0 such that $V_1 - V_1 \subset V_0$. Since the net $\{z_U\}_{U \in \mathbf{N}(0)}$ converges to 0 in X, without loss of generality we can assume $\{z_U\}_{U \in \mathbf{N}(0)} \subset V_1$. By $V_1 - V_1 \subset V_0$ and (15), for each $U \in \mathbf{N}(0)$, $s_U\theta_U \notin V_1$. By the boundedness of Θ, there is $\delta > 0$ such that $t\Theta \subset V_1$ whenever $0 < t < \delta$. Hence $s_U \geq \delta$ for all $U \in \mathbf{N}(0)$.

This and $z_U \to 0$ imply that there is $W \in \mathbf{N}(0)$ such that $z_W / s_W \in -U_1$. By (15),

$$b_W - x_0 = \frac{s_W}{r_W}\left(\frac{z_W}{s_W} - \theta_W\right) \in -S_{U_1}(\Theta) \qquad \textit{and} \qquad b_W - x_0 \neq 0.$$

This contradicts $x_0 \in E(A, S_{U_1}(\Theta))$.

The following results show that super efficiency has stability in some sense.

THEOREM 4.2. *Let X be a normed linear space, $S \subset X$ a closed convex cone with a base, and $A \subset X$ a closed set. Then for each $\bar{x} \in SE(A, S)$, there is $K > 0$ such that whenever A_1 is a weakly compact subset of X, $d(\bar{x}, E(A_1, S)) \le KH(A, A_1)$, where $H(A, A_1)$ is the Hausdorff distance between A and A_1.*

Proof. By $\bar{x} \in SE(A, S)$, there is $M > 0$ such that $\|a - \bar{x}\| \le M\|y\|$, whenever $a \in A$, $y \in X$, and $a - \bar{x} \le_S y$. Since A_1 is weakly compact, there is $z \in A_1$, such that

$$\|\bar{x} - z\| = d(\bar{x}, A_1) \le H(A, A_1).$$

Let $A_0 = A_1 \cap (z - S)$. Then A_0 is weakly compact. Hence $E(A_0, S) \neq \emptyset$. Take $\bar{y} \in E(A_0, S) \subset E(A_1, S)$. For each natural number n, there is $a_n \in A$ such that

$$\|\bar{y} - a_n\| \le \left(1 + \frac{1}{n}\right) d(\bar{y}, A) \le \left(1 + \frac{1}{n}\right) H(A, A_1).$$

By $z \ge_S \bar{y}$, $a_n - \bar{x} \le_S z - \bar{x} + a_n - \bar{y}$. Therefore,

$$\|a_n - \bar{x}\| \le M\big(\|z - \bar{x}\| + \|a_n - \bar{y}\|\big) \le M\left(2 + \frac{1}{n}\right) H(A, A_1).$$

Hence,

$$\begin{aligned} d\big(\bar{x}, E(A_1, S)\big) &\le \|\bar{x} - \bar{y}\| \le \|\bar{x} - a_n\| + \|a_n - \bar{y}\| \\ &\le \left(M\left(2 + \frac{1}{n}\right) + 1 + \frac{1}{n}\right) H(A, A_1). \end{aligned}$$

Let $n \to \infty$ and one obtains that

$$d\big(\bar{x}, E(A_1, S)\big) \le (2M + 1) H(A, A_1).$$

COROLLARY 4.1. *Let X be a normed linear space, $S \subset X$ a closed convex cone with a bounded base, and $A \subset X$ a closed set. Then for each $\bar{x} \in SE(A, S)$, there is $K > 0$ such that whenever A_1 is a weakly compact subset of X, $d(\bar{x}, SE(A_1, S)) \le KH(A, A_1)$.*

Proof. By Theorem 4.1,

$$\mathrm{cl}\big(SE(A_1, S)\big) \supset E(A_1, S) \supset SE(A_1, S).$$

Hence, $d(\bar{x}, SE(A_1, S)) = d(\bar{x}, E(A_1, S))$. By Theorem 4.2, we have

$$d\big(\bar{x}, SE(A_1, S)\big) \le KH(A, A_1).$$

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